Problem 5188. Given $\triangle ABC$ with coordinates $A(-5,0)$, $B(0,12)$ and $C(9,0)$. The triangle has an interior point P such that $\angle APB = \angle APC = 120^\circ$. Find the coordinates of point P.

Proposed by Kenneth Korbin, New York, NY

Solution by Ercole Suppa, Teramo, Italy

Let us construct equilateral triangles $\triangle ABD$, $\triangle AEC$ externally on the sides AB, AC of triangle $\triangle BAC$ and let denote by ω_1, ω_2 the circumcircles of $\triangle ABD$, $\triangle AEC$. The point P is the intersection point of ω_1, ω_2 different from O. In order to find the coordinates of D, E we use complex numbers. If we denote respectively by $a = -5$, $b = 12i$, $c = 9$ the affixes of A, B, C we get:

$$\begin{aligned} d &= a + (b-a)e^{\frac{\pi}{3}i} = \frac{-5-12\sqrt{3}}{2} + \frac{12+5\sqrt{3}}{2}i \\ e &= a + (c-a)e^{\frac{\pi}{3}i} = 2 - 7\sqrt{3}i \end{aligned}$$

so the coordinates of D, E are $D\left(\frac{-5-12\sqrt{3}}{2}, \frac{12+5\sqrt{3}}{2}\right)$ and $E\left(2, -7\sqrt{3}\right)$.

The equations of ω_1, ω_2 are:

$$\begin{aligned} \omega_1 &: 169\sqrt{3}x^2 + 169\sqrt{3}y^2 + \left(2028 + 845\sqrt{3}\right)x + \left(-845 - 2028\sqrt{3}\right)y + 10140 = 0 \\ \omega_2 &: 196\sqrt{3}x^2 + 196\sqrt{3}y^2 - 784\sqrt{3}x + 2744y - 8820\sqrt{3} = 0 \end{aligned}$$

and, after some ugly calculations, we get

$$P = \omega_1 \cap \omega_2 = \left(-\frac{2\left(-981 + 112\sqrt{3}\right)}{2353}, -\frac{21\left(-896 + 263\sqrt{3}\right)}{2353}\right)$$

$\square$